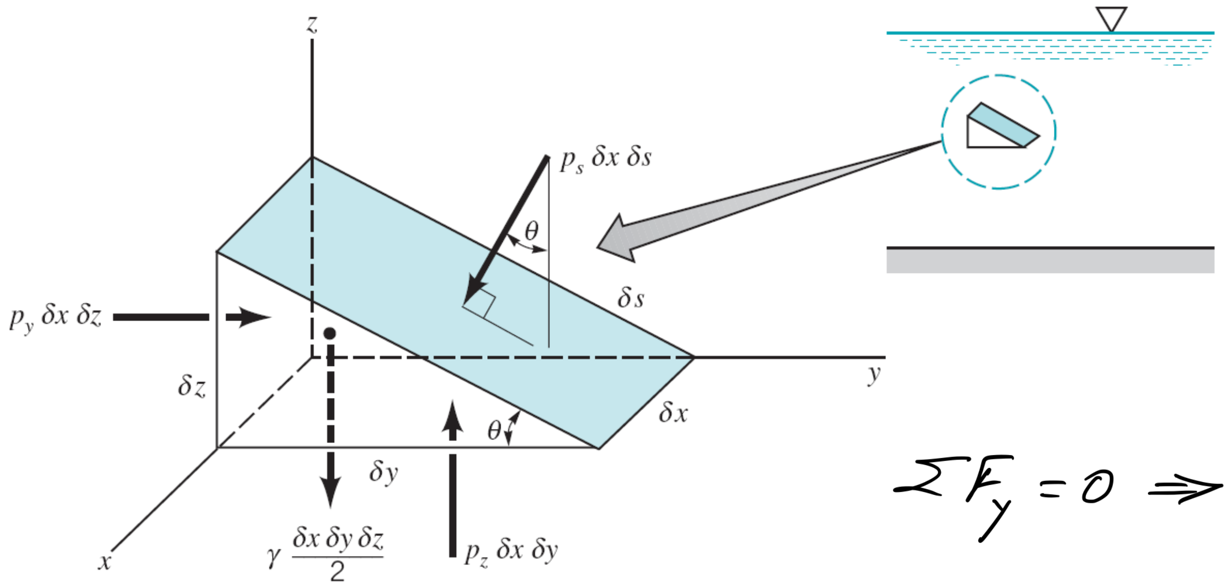
Fluid statics

Fluids at rest: no shearing forces



Forces on an arbitrary wedge-shaped element of fluid.

$$\Sigma F_y = 0 \Rightarrow P_y\,\delta_x\,\delta_z = P_s\,\delta_s\cdot\delta_x\cdot\sin\theta$$

$$\Rightarrow P_y\,\delta_z = P_s\cdot\underbrace{\delta_s\cdot\sin\theta}_{\delta_z}$$

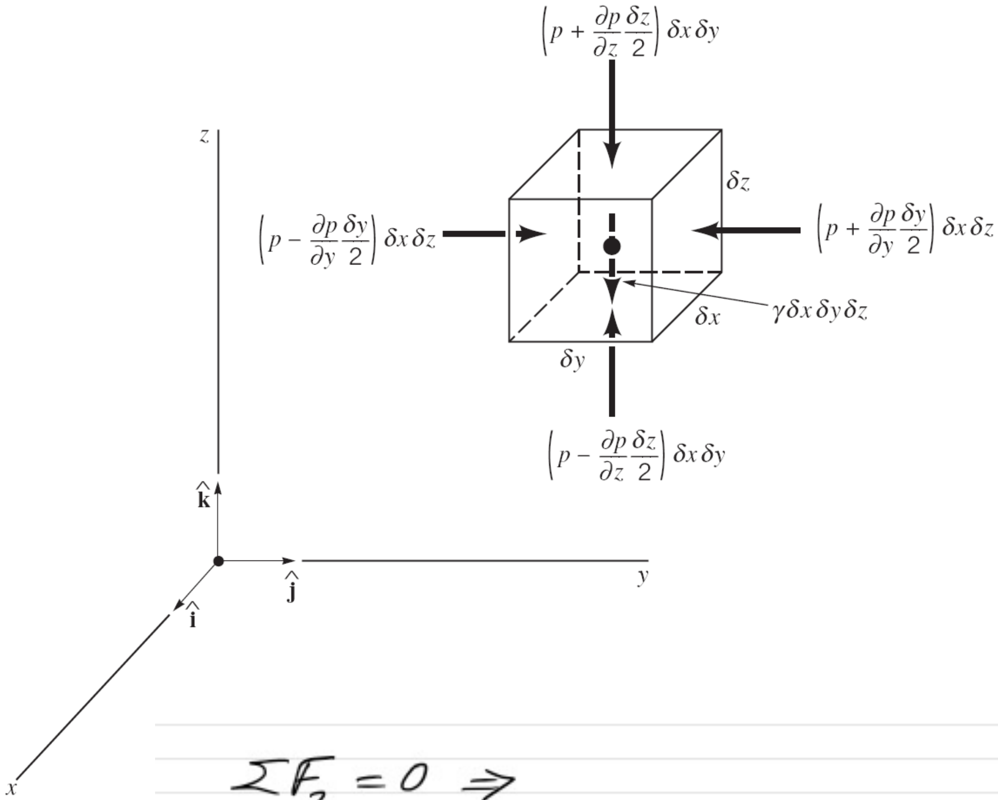
$$\Rightarrow P_y = P_s$$

Since θ is arbitrary, this means that:

Pressure at a point is independent of direction

Pressure variation from point to point

Surface and body forces acting on small fluid element.



$$\sum F_z = 0 \Rightarrow$$

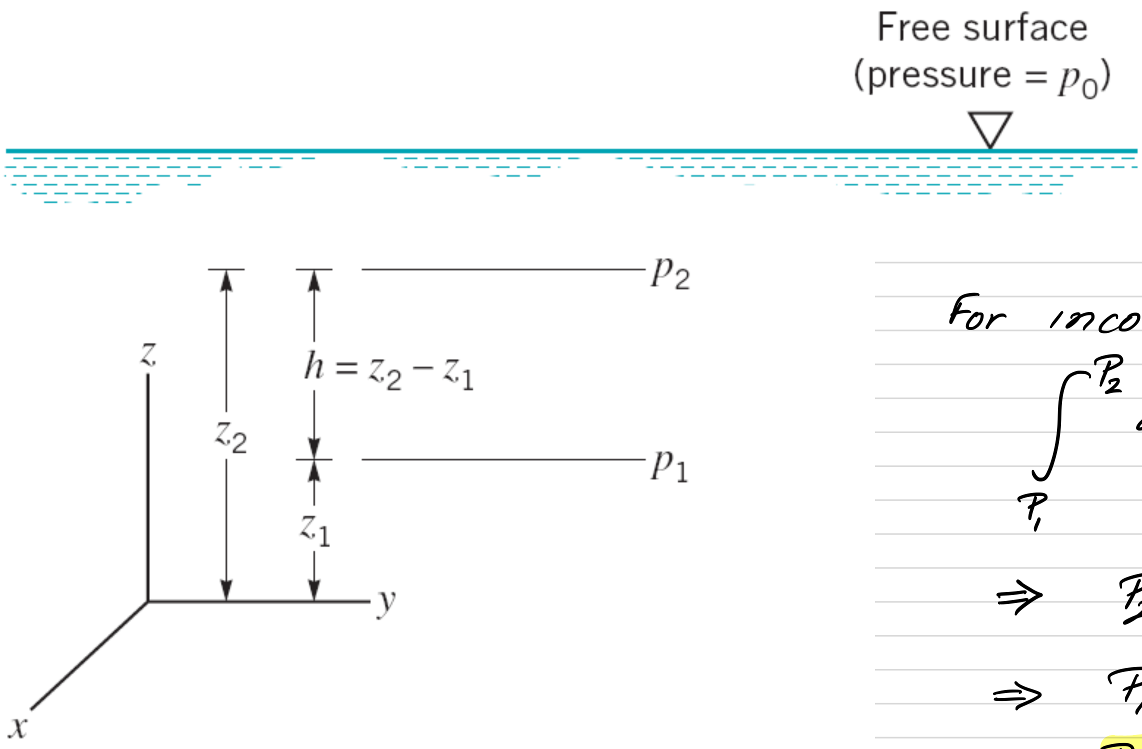
$$\Rightarrow \left(p - \frac{\partial p}{\partial z}\frac{\delta z}{2}\right)(\delta x\,\delta y)$$

$$- \gamma \cdot \delta x \cdot \delta y \cdot \delta z - \left(p + \frac{\partial p}{\partial z}\frac{\delta z}{2}\right)(\delta x \cdot \delta y) = 0$$

$$\Rightarrow \quad -\frac{\partial p}{\partial z} = \gamma \qquad \text{also } p \neq f(x,y)$$

$$\therefore \quad \frac{dp}{dz} = -\gamma$$

Hydrostatic pressure variation



For incompressible fluid ($\gamma = ct$)

$$\int_{P_1}^{P_2} dp = -\gamma \int_{z_1}^{z_2} dz =$$

$$\Rightarrow \quad P_2 - P_1 = -\gamma(z_2 - z_1)$$

$$\Rightarrow \quad P_1 - P_2 = \gamma(z_2 - z_1)$$

$$P_1 - P_2 = \gamma h \quad \text{or} \quad P_1 = P_2 + \gamma h$$

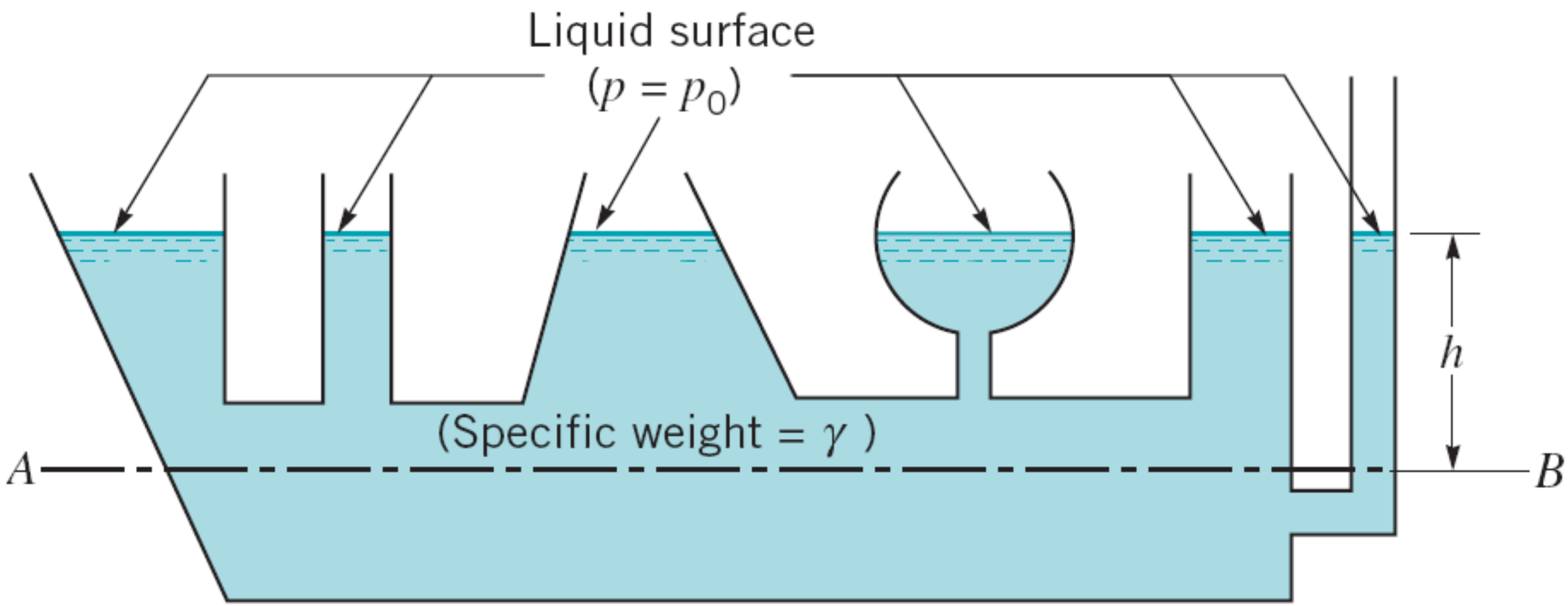
Hydrostatic pressure variation

Concept of "pressure head"

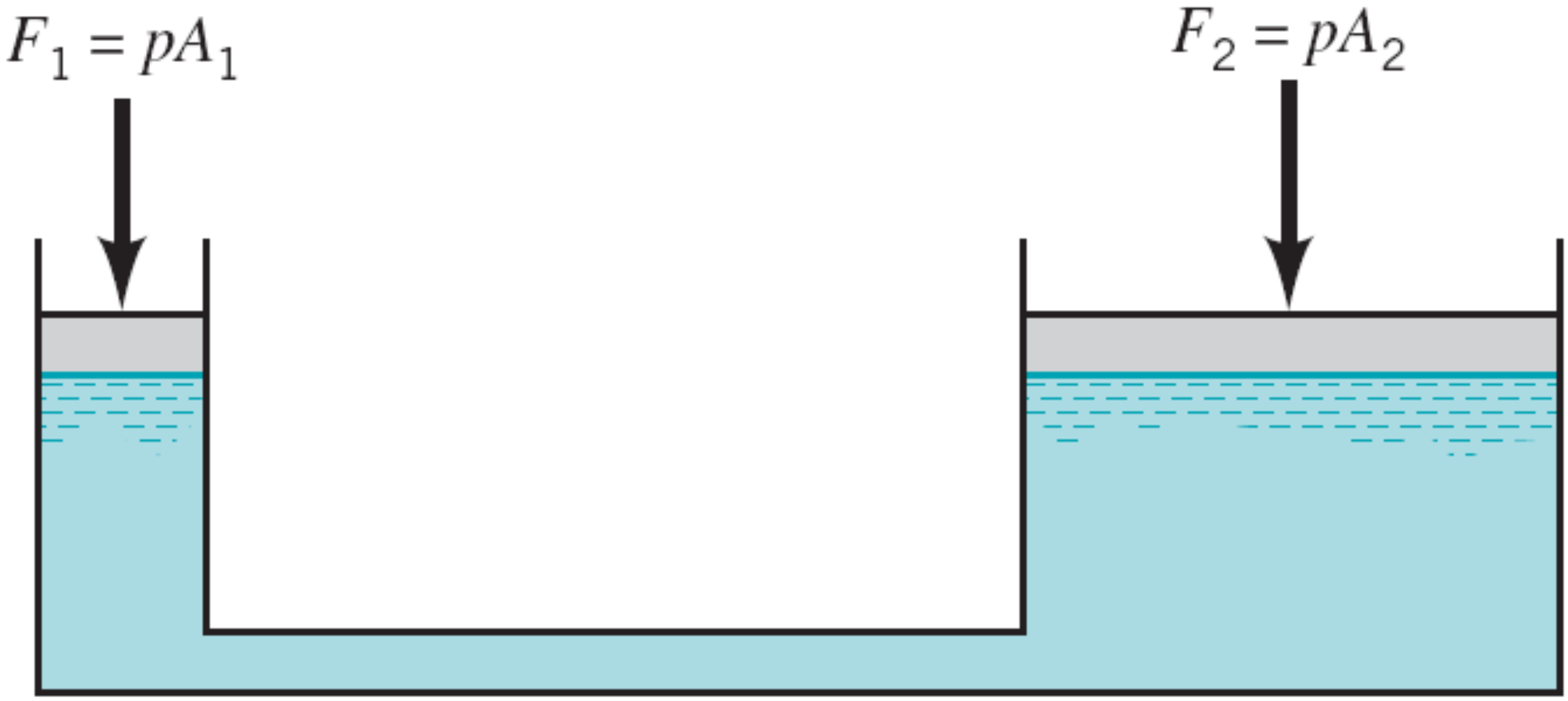
$$h = \frac{P_1 - P_2}{\gamma}$$

In general $\quad P = P_0 + \gamma h$

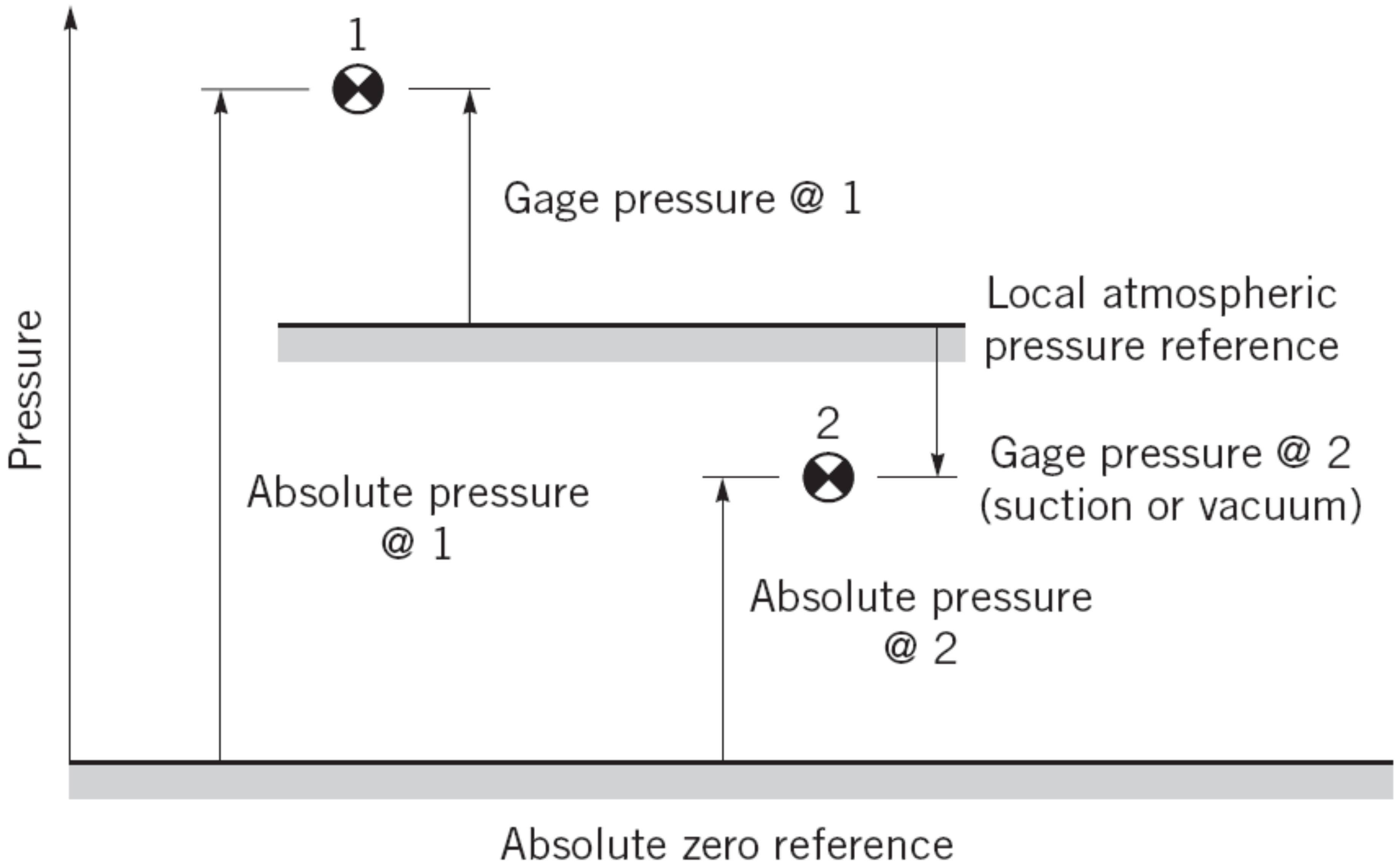
Fluid equilibrium in a container of arbitrary shape



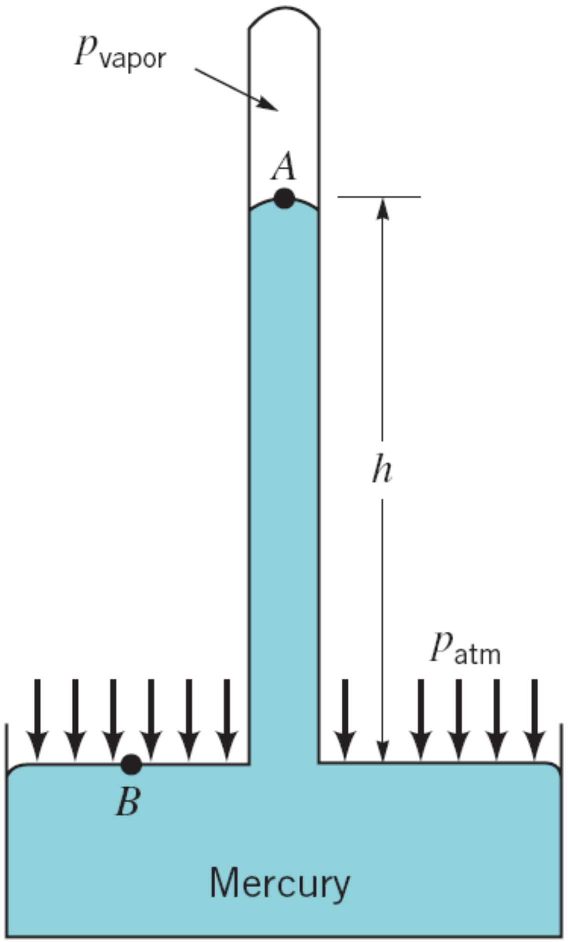
Transmission of fluid pressure



Graphical representation of gage and absolute pressure



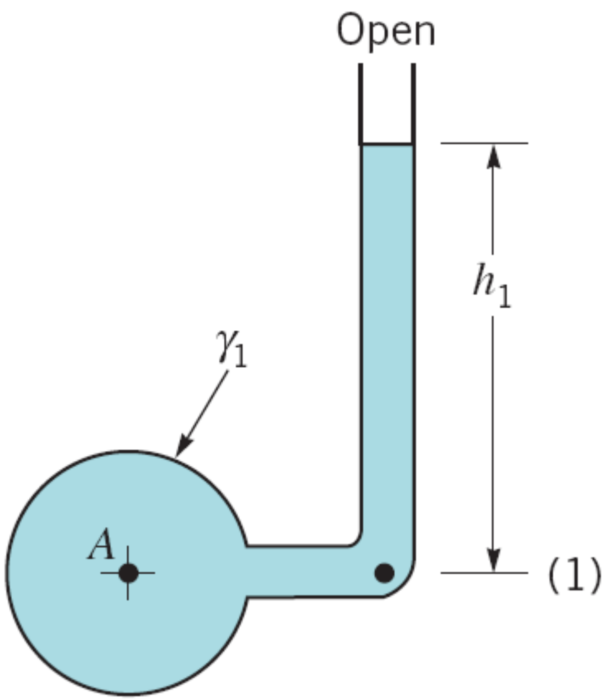
Mercury barometer



$$P_{atm} = \underbrace{\cancel{P_{vapor}}}_{\approx \phi} + \gamma_{Hg} \cdot h$$

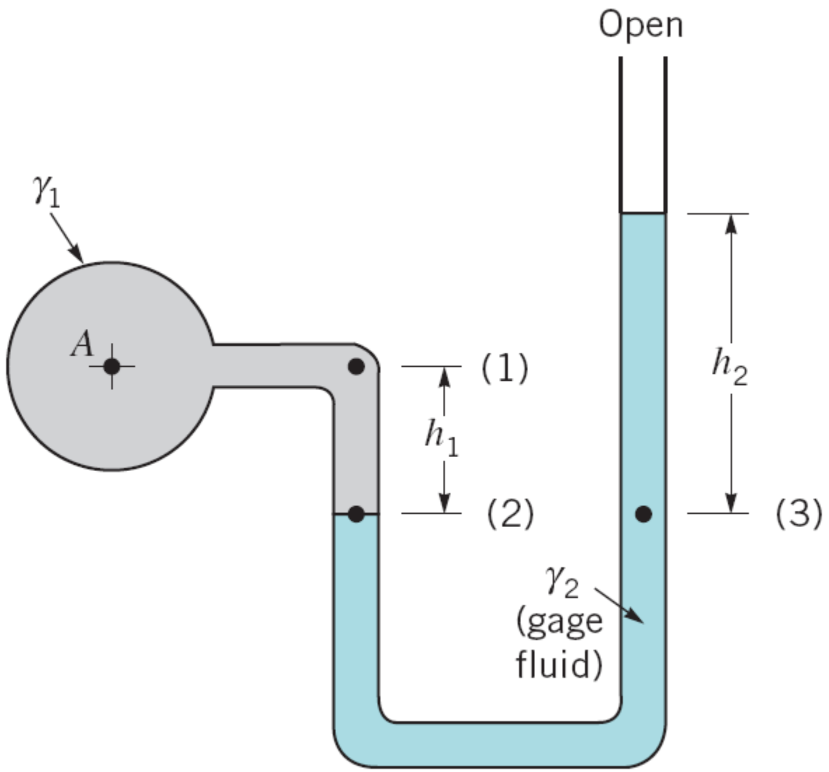
$$\Rightarrow \quad P_{atm} = \gamma_{Hg} \cdot h$$

Pressure measurements: piezometer tube



$$p_A = \gamma_1 \cdot h_1$$

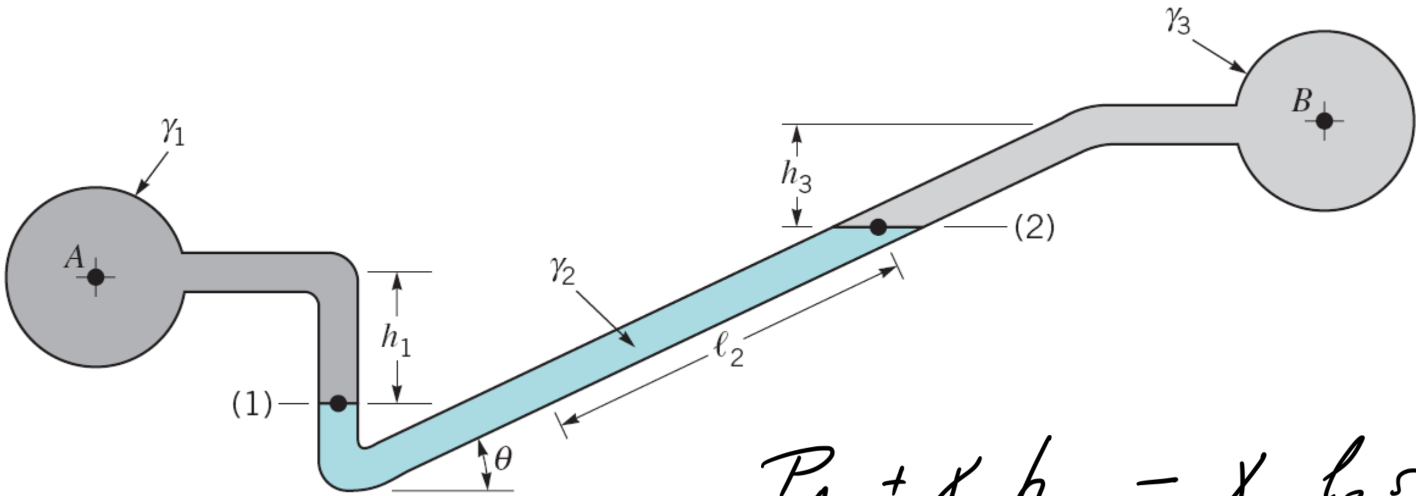
Simple U-tube manometer.



$$P_A + \gamma_1 \cdot h_1 - \gamma_2 \cdot h_2 = \cancel{P_{atm}} \rightarrow \emptyset \text{ (gage pressure)}$$

$$\Rightarrow P_A = \gamma_2 h_2 - \gamma_1 h$$

Inclined-tube manometer



$$P_A + \gamma_1 h_1 - \gamma_2 \ell_2 \sin\theta - \gamma_3 \cdot h_3 = P_B$$

$$\Rightarrow P_A - P_B = \gamma_2 \ell_2 \sin\theta + \gamma_3 h_3 - \gamma_1 h_1$$

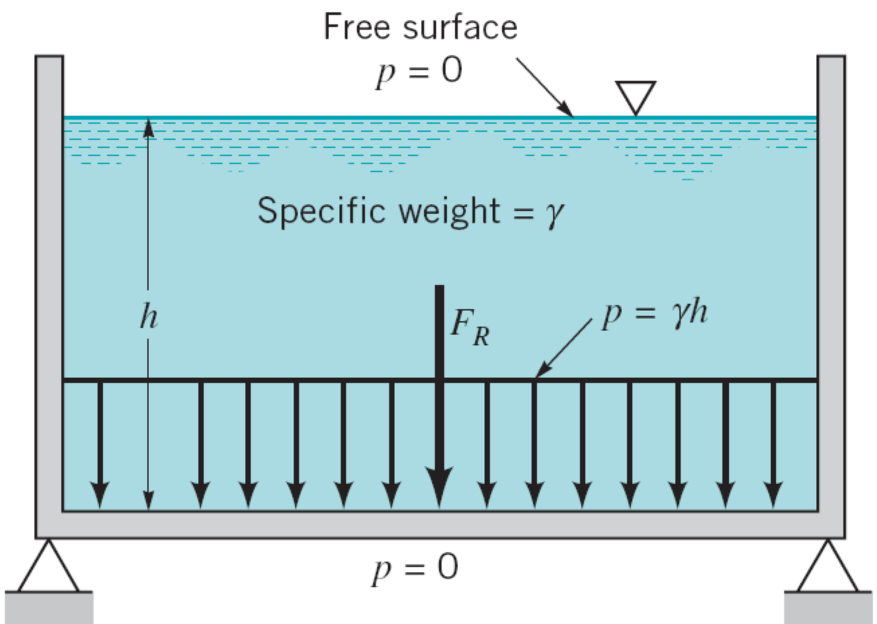
If gas in A and B $\quad \gamma_1, \gamma_3 \ll \gamma_2$

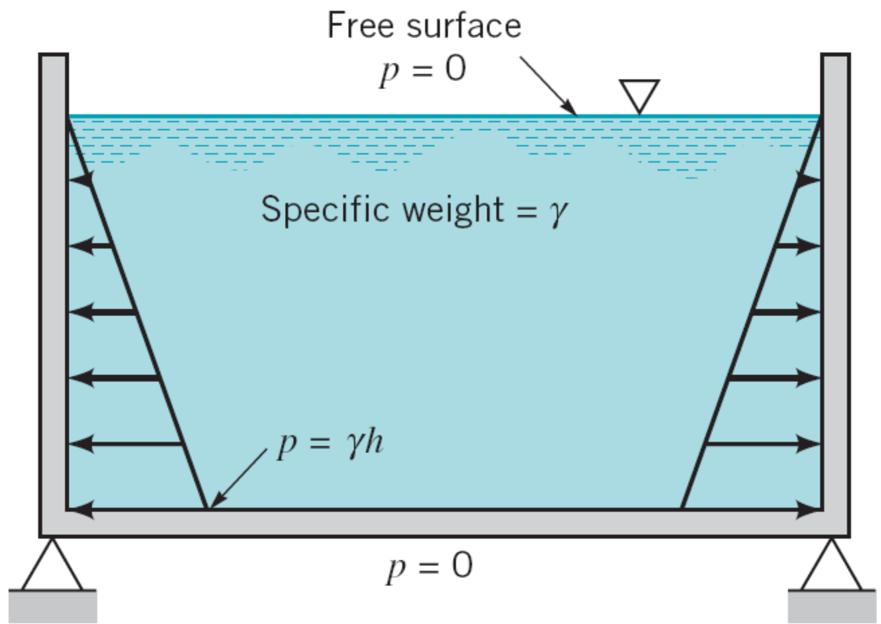
then, $\quad P_A - P_B = \gamma_2 \ell_2 \sin\theta$

$$\ell_2 = \frac{P_A - P_B}{\gamma_2 \sin\theta}$$

Note: as $\theta \to 0 \rightsquigarrow \ell_2 \to \infty$

Hydrostatic force on a plane surface



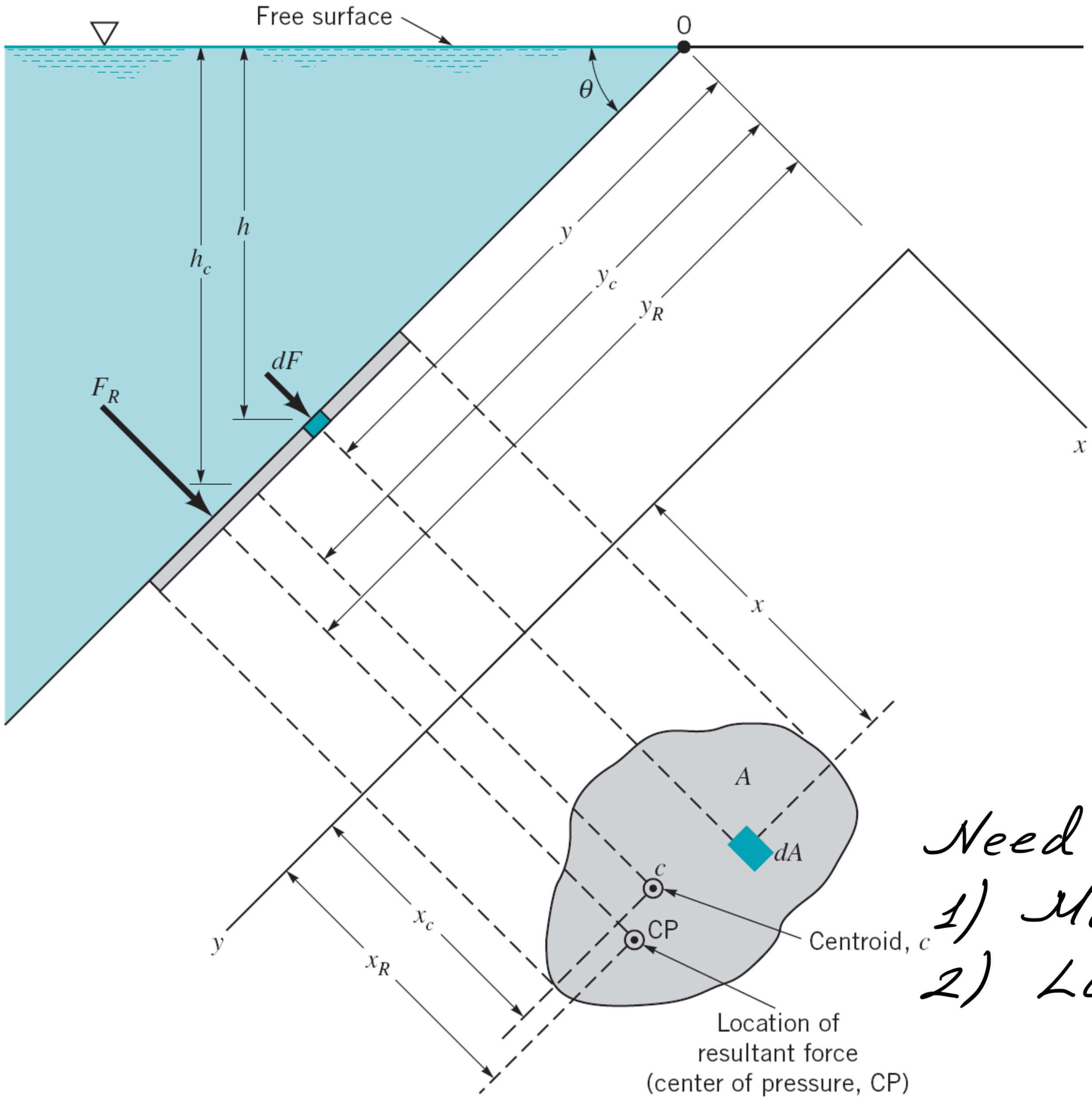


- No shear forces ⇒ forces are perpendicular to the surface

- When pressure is <u>uniform</u> (horizontal plane)

$$F_R = P \cdot A = \gamma \cdot h \cdot A$$

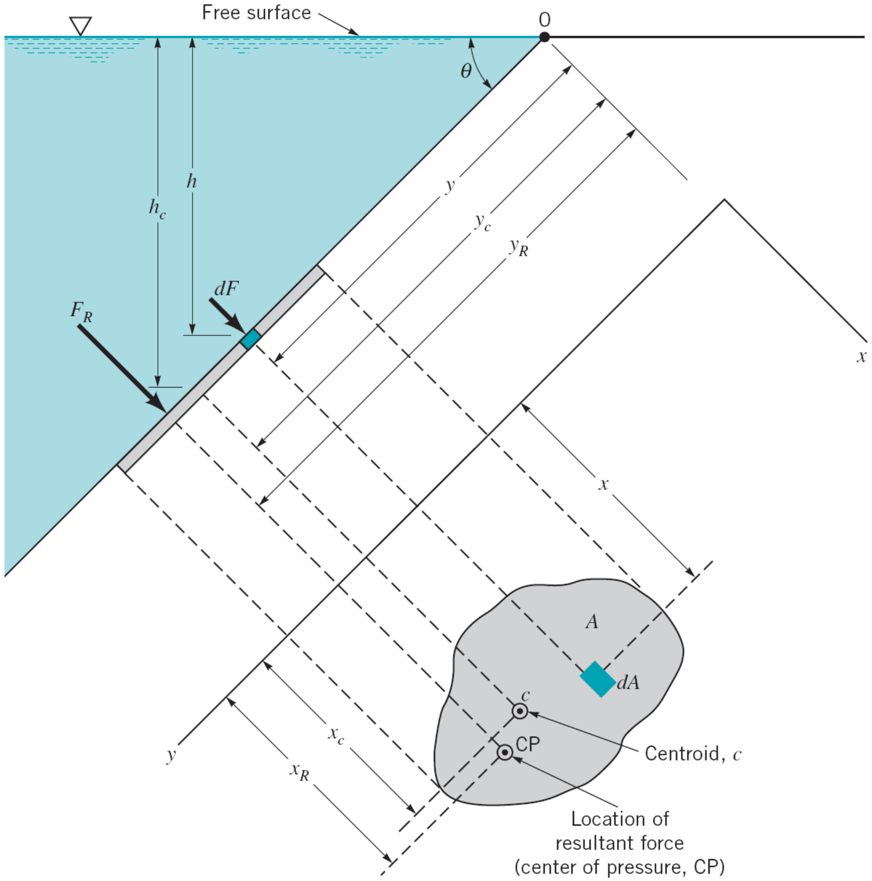
Hydrostatic force on an inclined plane surface



Need to define two things:

1) Magnitude of F_R
2) Location of F_R

Hydrostatic force on an inclined plane surface (2)



Force acting on the differential element

$$dF = p \cdot dA = \gamma \cdot h \cdot dA$$

The magnitude of the resultant force is:

$$F_R = \int_A \gamma \cdot h \cdot dA = \int_A \gamma \cdot y \cdot \sin\theta \cdot dA$$

For $\gamma = ct$ and $\theta = ct$

$$F_R = \gamma \cdot \sin\theta \underbrace{\int_A y \cdot dA}_{\text{1st moment of area wrt the x-axis}}$$

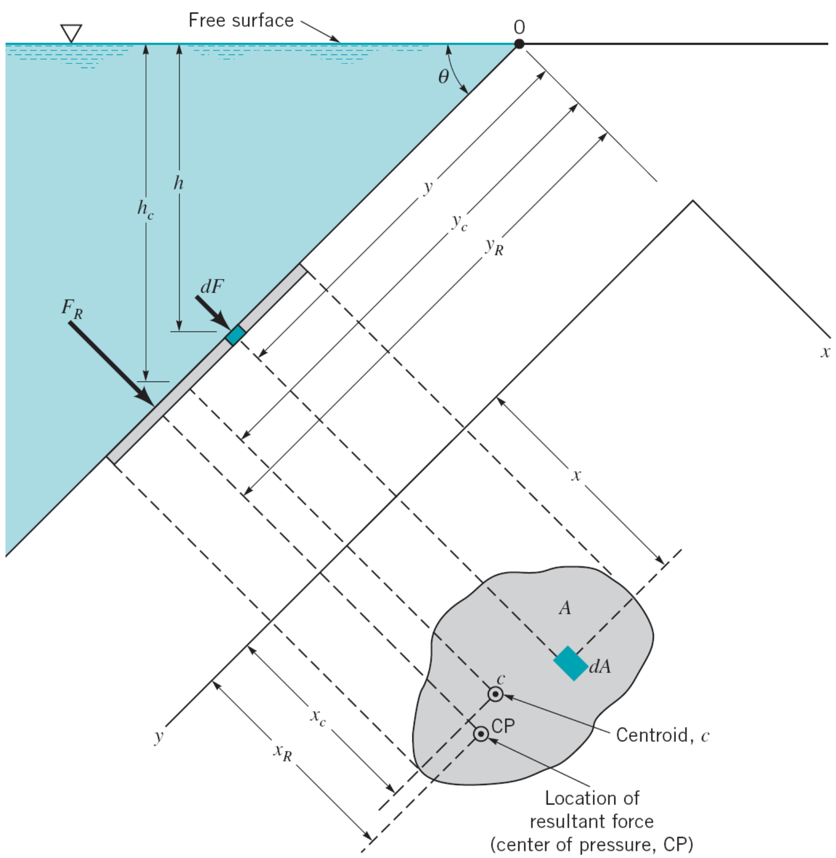
Recall: $Y_c = \dfrac{\int_A y\,dA}{A}$ where y_c = distance to centroid

Hence, $$Y_c \cdot A = \int_A y\,dA$$

$$F_R = \gamma \underbrace{\sin\theta \, Y_c}_{h_c} \cdot A \Rightarrow F_R = \gamma \cdot h_c \cdot A$$

acting $\perp$ to the surface

Hydrostatic force on an inclined plane surface (3)



Determine the location of F_R

Moments about x-axis of F_R and individual pressure forces must be equal

$$F_R \cdot y_R = \int_A y \, dF = \int_A y \cdot \gamma \cdot \sin\theta \cdot y \, dA$$

$$\Rightarrow y_R = \frac{\gamma \cdot \sin\theta \cdot \int_A y^2 \cdot dA}{F_R} = \frac{\gamma \cdot \sin\theta \cdot \int_A y^2 \cdot dA}{\gamma \cdot A \cdot y_c \sin\theta}$$

$$\Rightarrow y_R = \frac{\int_A y^2 \cdot dA}{y_c \cdot A}$$

2nd moment of inertia wrt x-axis, I_x

$$\Rightarrow y_R = \frac{I_x}{y_c \cdot A}$$

For x-axis passing through the centroid

Parallel axes theorem: $I_x = I_{xc} + A y_c^2$

$$\therefore \quad y_R = \frac{I_{xc}}{y_c A} + y_c$$

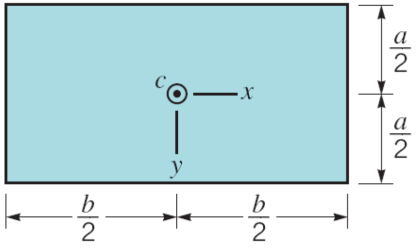
Note: y_R always $\geq y_c$

Similarly,

$$x_R = \frac{I_{xyc}}{y_c \cdot A} + x_c$$

Note: For symmetrical planes $I_{xyc} = 0 \Rightarrow x_R = x_c$

Geometric properties of some common shapes



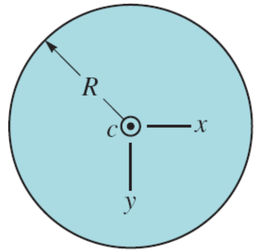
(*a*) Rectangle

$$A = ba$$

$$I_{xc} = \frac{1}{12} ba^3$$

$$I_{yc} = \frac{1}{12} ab^3$$

$$I_{xyc} = 0$$

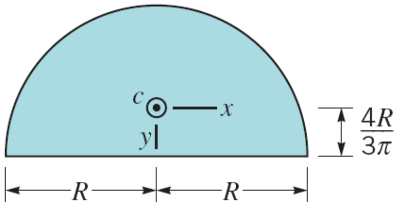


(*b*) Circle

$$A = \pi R^2$$

$$I_{xc} = I_{yc} = \frac{\pi R^4}{4}$$

$$I_{xyc} = 0$$



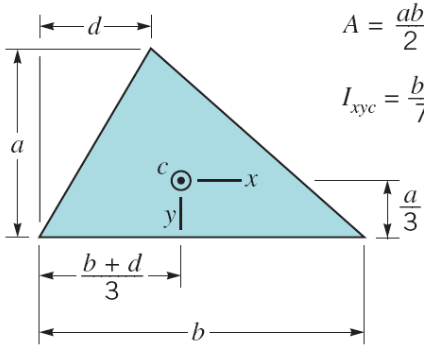
(*c*) Semicircle

$$A = \frac{\pi R^2}{2}$$

$$I_{xc} = 0.1098R^4$$

$$I_{yc} = 0.3927R^4$$

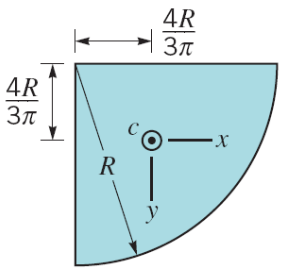
$$I_{xyc} = 0$$



(*d*) Triangle

$$A = \frac{ab}{2} \qquad I_{xc} = \frac{ba^3}{36}$$

$$I_{xyc} = \frac{ba^2}{72}(b - 2d)$$



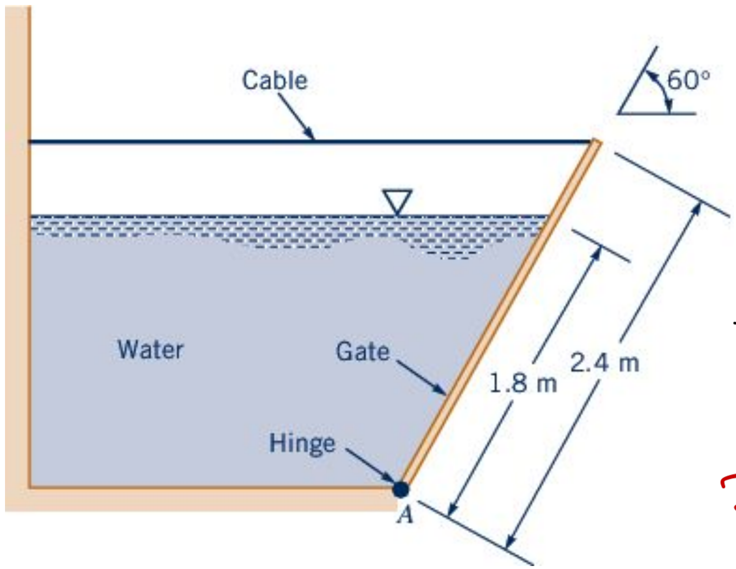
(*e*) Quarter circle

$$A = \frac{\pi R^2}{4}$$

$$I_{xc} = I_{yc} = 0.05488R^4$$

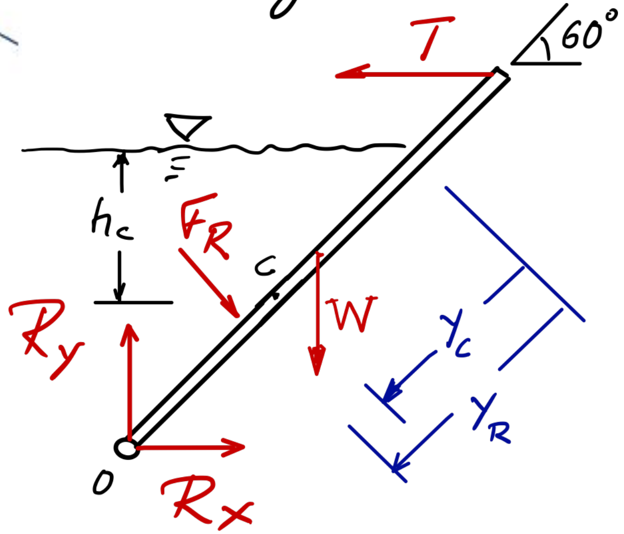
$$I_{xyc} = -0.01647R^4$$

Example: force on an inclined gate



The gate is 1.2 m wide and weighs 3.6 kN.
What is the tension, T, in the cable?

First, let's consider the free body diagram (FBD) of the gate:



$$A = 1.8\,m \times 1.2\,m = \underline{2.16\,m^2}$$

$$h_c = Y_c \cdot \sin 60 = 0.9\,m \cdot \sin 60$$

$$\Rightarrow \quad h_c = \underline{0.78\,m}$$

$$F_R = \gamma \cdot h_c \cdot A$$

$$= 9800\,N/m^3 \cdot 0.78\,m \times 2.16\,m^2$$

$$\Rightarrow \quad \underline{F_R = 16.5\ kN}$$

To locate F_R:

$$Y_R = \frac{I_{xc}}{Y_c A} + Y_c = \frac{\frac{1}{12} \cdot 1.2 \times 1.8^3}{0.9 \times 2.16}\,m + 0.9\,m$$

$$\Rightarrow \quad \underline{Y_R} = 0.3\,m + 0.9\,m = \underline{1.2\,m}$$

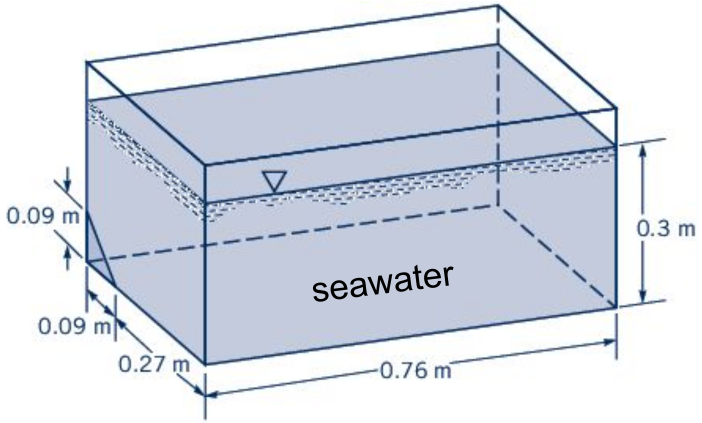
For equilibrium: $\Sigma M_0 = 0$ (+)

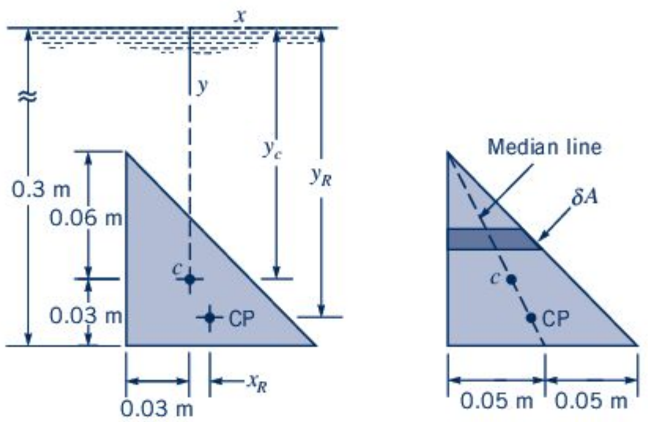
$$\Rightarrow \quad T \cdot 2.4\,m \cdot \sin 60^\circ - W \cdot \frac{2.4\,m}{2} \cos 60^\circ - F_R \cdot (1.8 - Y_R) = 0$$

$$\Rightarrow \quad T = \frac{3.6\,kN \cdot 1.2\,m \cdot \cos 60 + 16.5\,kN \cdot (1.8 - 1.2)\,m}{2.4\,m \cdot \sin 60}$$

$$\Rightarrow \quad \underline{T = 6.84\ kN}$$

Hydrostatic Pressure Force on a plane triangular surface





$$F_R = \gamma \cdot h_c \cdot A = (10.1\ kN/m^3)\cdot(0.27\,m)\cdot\left(\tfrac{1}{2}\ 0.9 \times 0.9\ m^2\right)$$

$$\Rightarrow \underline{\underline{F_R = 11\ N}}$$

The y-coordinate of the center of pressure, Y_c:

$$Y_R = \frac{I_{xc}}{Y_c A} + Y_c$$

where $$I_{xc} = \frac{(0.09\,m)\times(0.09\,m)^3}{36} = 1.82\times10^{-6}\,m^4$$

so that $$Y_R = \frac{1.82\times10^{-6}\,m^4}{(0.27\,m)\cdot\left(\tfrac{1}{2}\ 0.09^2\,m^2\right)} + 0.27\,m$$

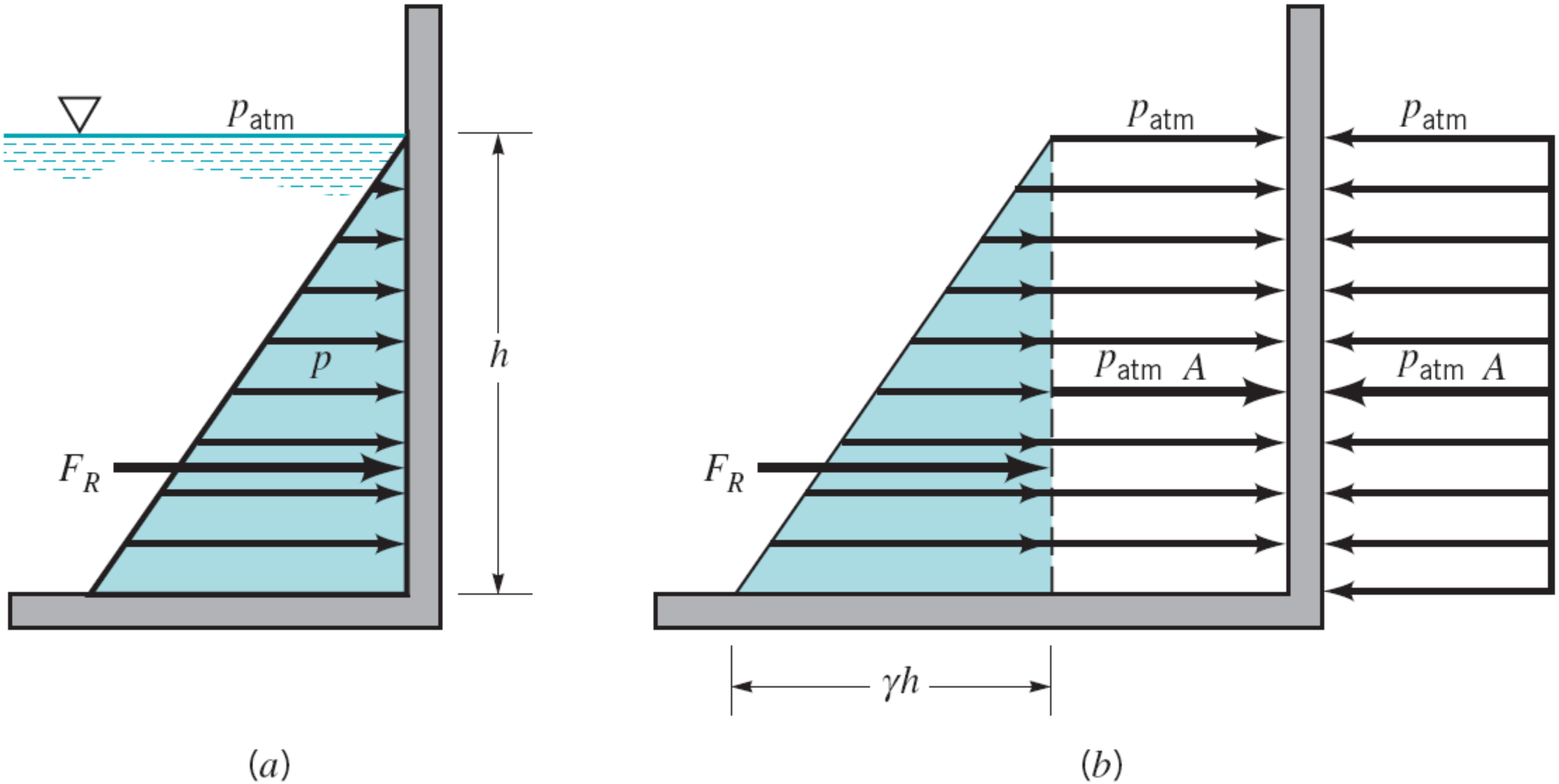
$$\Rightarrow Y_R = 0.0017\,m + 0.27\,m = \underline{\underline{0.272\,m}}$$

Similarly, $$x_R = \frac{I_{xyc}}{Y_c \cdot A} + x_c$$

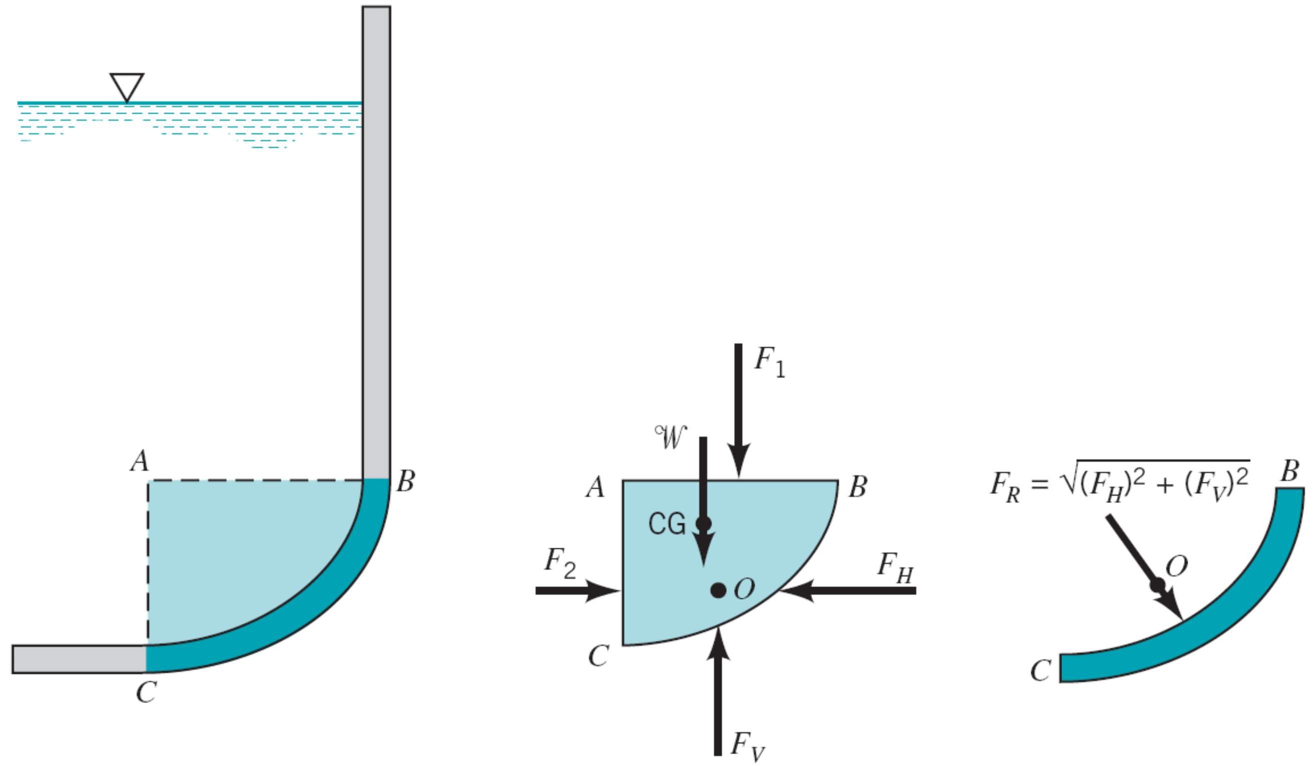
where $$I_{xyc} = \frac{(0.09\,m)\cdot(0.09\,m)^3}{72} = 9.11\times10^{-7}\,m^4$$

Hence, $$x_R = \frac{9.11\times10^{-7}\,m^4}{(0.27\,m)\cdot\left(\tfrac{1}{2}\ 0.09^2\,m^2\right)} = \underline{\underline{8.38\times10^{-4}\,m}}$$

Effect of atmospheric pressure on the resultant force acting on a plane vertical wall



Hydrostatic force on a curved surface.



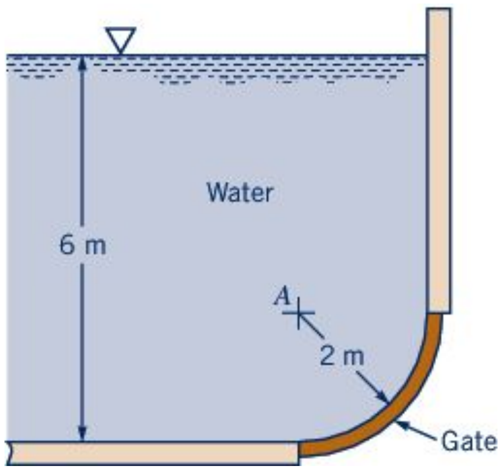
For equilibrium: $F_H = F_2$

$F_V = F_1 + W$

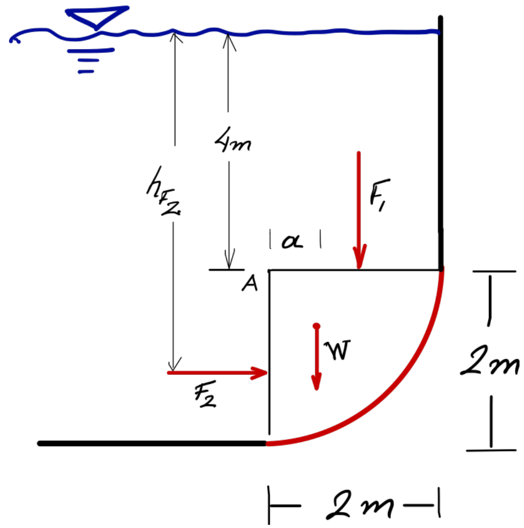
The resultant force is $F_R = \sqrt{F_H^2 + F_V^2}$

To locate the point of application O, take moments about a convenient axis.

Example: hydrostatic force on a curved surface.



width = 3m



$$a = \frac{4R}{3\pi} = \frac{4 \times 2m}{3\pi} = \underline{\underline{0.849\ m}}$$

$$F_1 = \gamma\, h_{c_1} \cdot A_1$$
$$= 9.8\ kN/m^3 \cdot 4m \cdot (2\times3)m^2 = \underline{\underline{235.2\ kN}}$$

$$F_2 = \gamma \cdot h_{c_2} \cdot A_2$$
$$= 9.8\ kN/m^3 \cdot 5m \cdot (2\times3)\,m^2 = \underline{\underline{294\ kN}}$$

$$h_{F_2} = h_{c_2} + \frac{I_{xx_2}}{h_{c_2} \cdot A_2} = 5 + \frac{\frac{1}{12}\,3\,2^3}{5 \cdot (2\times3)} = 5 + \frac{1}{15} = \underline{\underline{5.067m}}$$

$$W = \gamma \cdot Volume = 9.8\ kN/m^3 \cdot \left(\frac{1}{4}\pi 2^2 m^2\right) \cdot 3m = \underline{\underline{92.4\ kN}}$$

$$F_H = F_2 = \underline{\underline{294\ kN}}$$

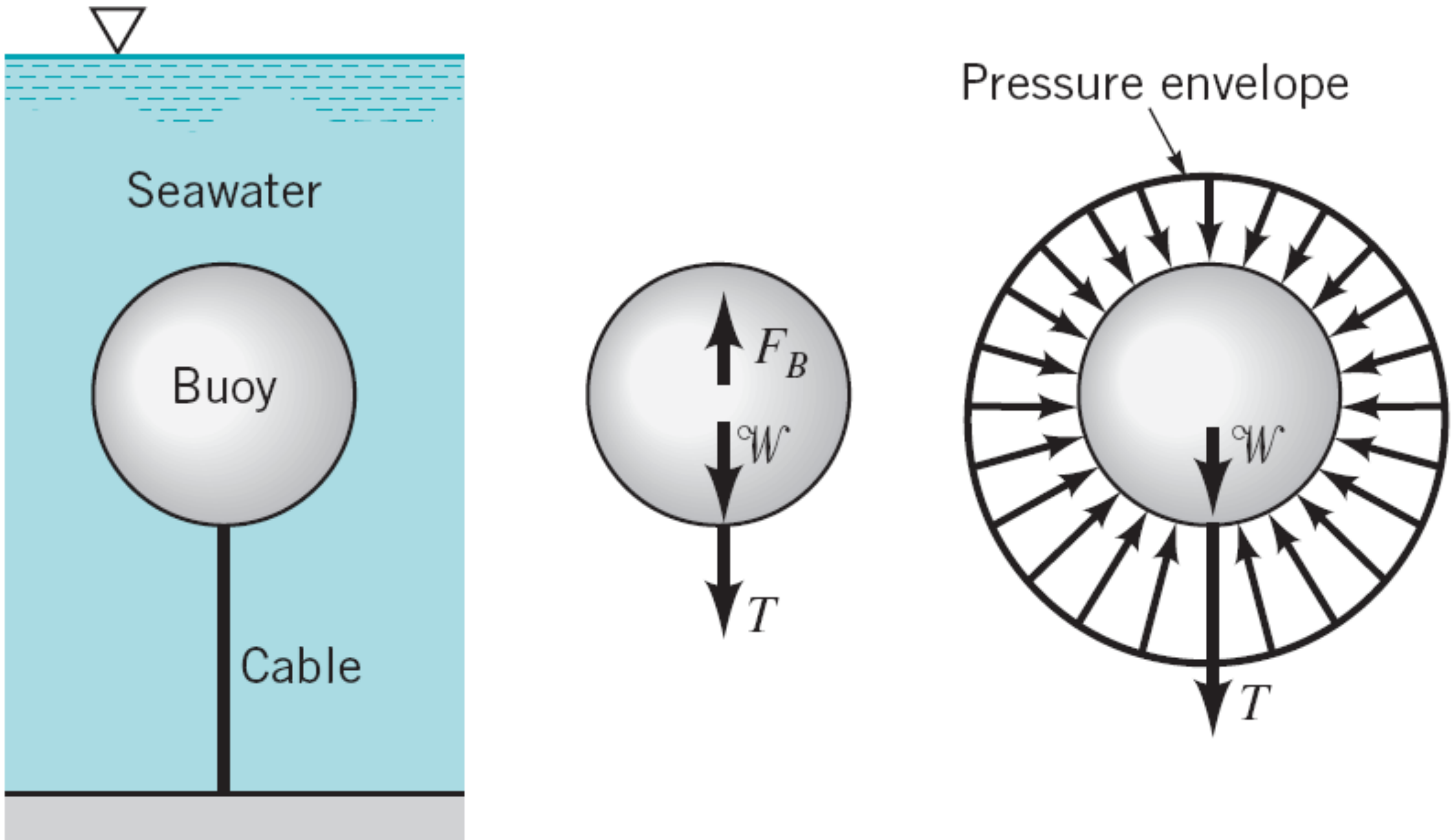
$$F_V = F_1 + W = (235.2 + 92.4)\ kN = \underline{\underline{328\ kN}}$$

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{294^2 + 328^2}\ kN = \underline{\underline{440\ kN}}$$

The resultant force will pass through point A.

Why?

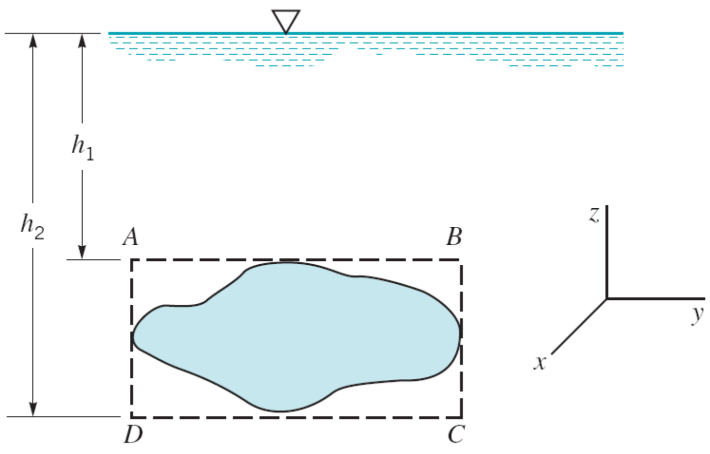
Buoyancy



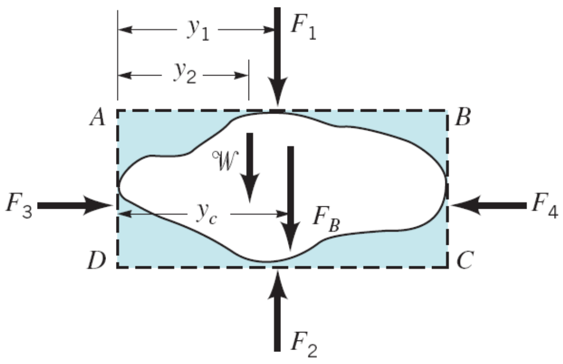
(a) (b) (c)

- Pressure distribution around a body increases with depth
- Creates an upward force: **Buoyant force, F_B**

Archimedes' principle



(a)



(b)

W = weight of the fluid in ABCD
V_f = volume of the fluid in ABCD
V_b = volume of the submerged body

Consider the FBD of block ABCD, which includes <u>only the fluid part</u>

$$\Sigma F_y = 0 \Rightarrow F_3 = F_4$$

$$\Sigma F_z = 0 \Rightarrow F_2 = W + F_B + F_1$$

$$\Rightarrow F_B = F_2 - F_1 - W$$

$$= \gamma_f \cdot h_2 \cdot A - \gamma_f \cdot h_1 \cdot A - \gamma_f V_f$$

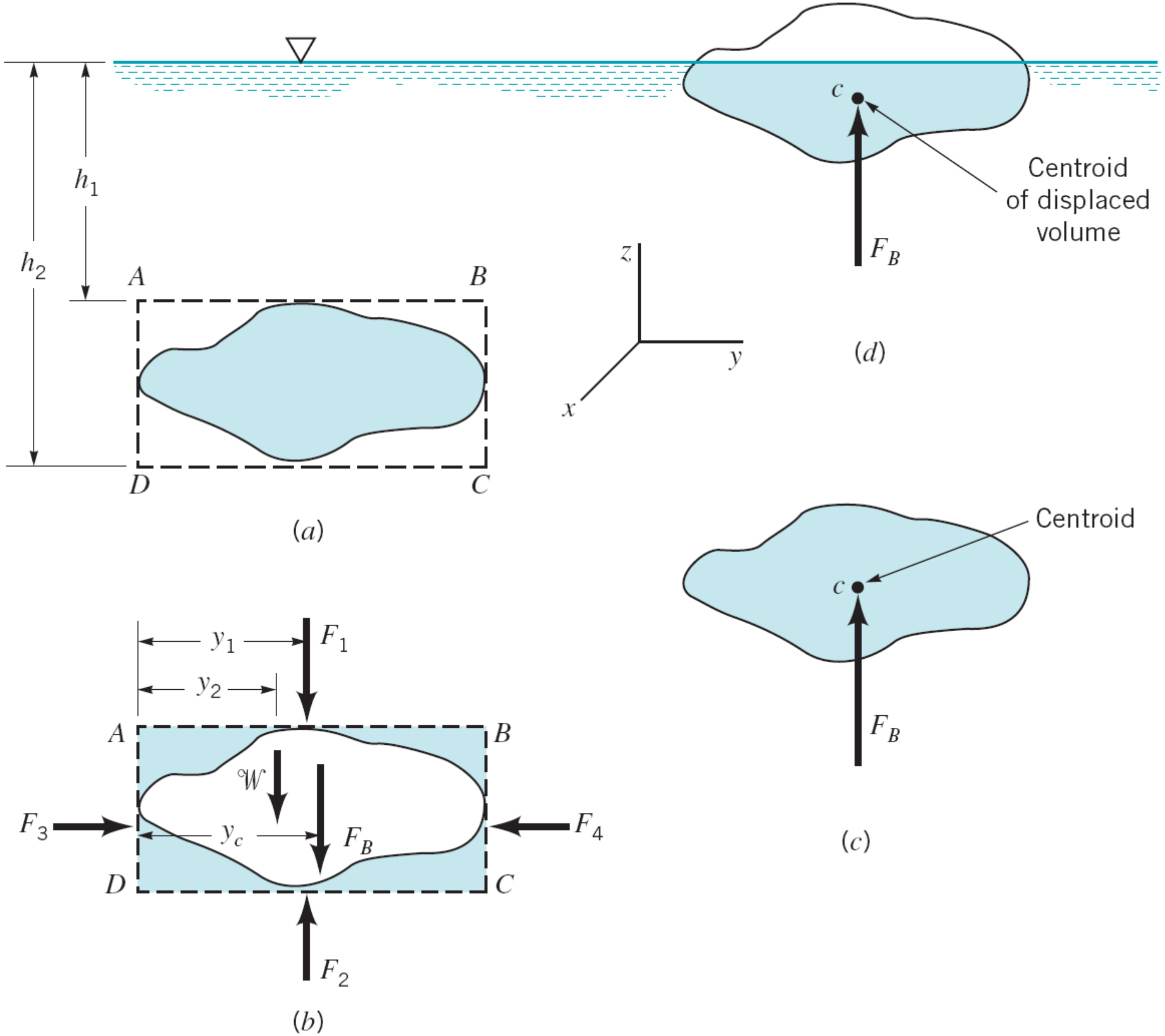
$$= \gamma_f \underbrace{(h_2 - h_1) \cdot A}_{V_f + V_b} - \gamma_f \cdot V_f$$

$$= \gamma_f (V_f + V_b) - \gamma_f \cdot V_f$$

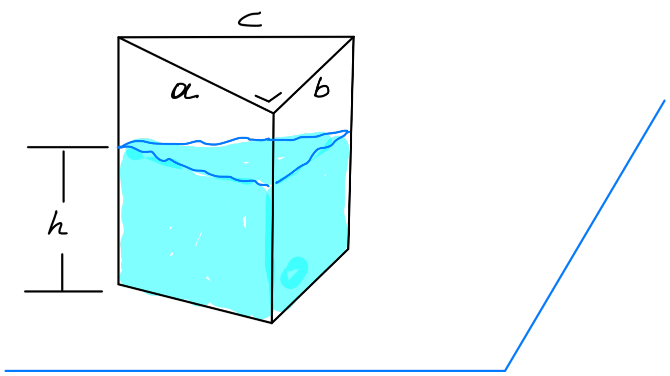
$$\Rightarrow \boxed{F_B = \gamma_F \cdot V_b}$$

F_B passes through the <u>centroid</u> of the <u>displaced volume</u>

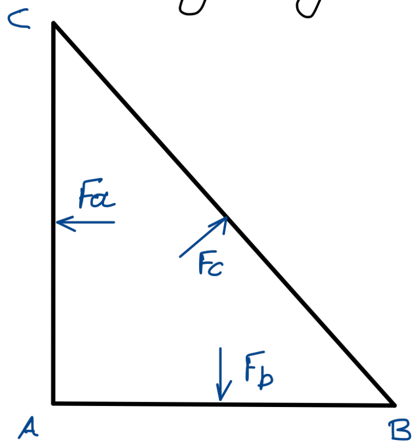
Archimedes' principle



Pythagora’s theorem



Free body diagram on the vase:



$$F_a = \gamma \cdot \frac{h}{2}(h \cdot a) = \frac{1}{2}\gamma h^2 a$$

$$F_b = \frac{1}{2}\gamma h^2 b$$

$$F_c = \frac{1}{2}\gamma h^2 c$$

For equilibrium: $\Sigma \vec{F} = 0$ and $\Sigma \overset{\curvearrowleft}{M_i} = 0$

choose $\Sigma M_C = 0$ (why?)

$$-F_a \times \frac{a}{2} - F_b \times \frac{b}{2} = F_c \times \frac{c}{2}$$

$$\Longrightarrow F_c c = F_a a + F_b b$$

$$\Longrightarrow \frac{1}{2}h^2c^2 = \frac{1}{2}h^2a^2 + \frac{1}{2}h^2b^2$$

$$\boxed{\Longrightarrow c^2 = a^2 + b^2}$$ Pythagora’s theorem